iopscience.iop.org

Home Search Collections Journals About Contact us My IOPscience

Towards classification of $(2+1)$-dimensional integrable equations. Integrability conditions I

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

1998 J. Phys. A: Math. Gen. 31 6707

(http://iopscience.iop.org/0305-4470/31/31/015)

View the table of contents for this issue, or go to the journal homepage for more

Download details:
IP Address: 171.66.16.102
The article was downloaded on 02/06/2010 at 07:09

Please note that terms and conditions apply.

J. Phys. A: Math. Gen. **31** (1998) 6707–6715. Printed in the UK PII: S0305-4470(98)92996-1

Towards classification of $(2+1)$-dimensional integrable equations. Integrability conditions I

A V Mikhailov†‡ and R I Yamilov§

† Applied Mathematics Department, University of Leeds, Leeds LS2 9JT, UK
‡ Landau Institute for Theoretical Physics, Russian Academy of Sciences, 2 Kosygina Street, Moscow 117940, Russia
§ Ufa Institute of Mathematics, Russian Academy of Sciences, 112 Chernyshevsky Street, Ufa 450000, Russia

Received 2 April 1998

Abstract. In this paper we attempt to extend the symmetry approach (well developed in the case of $(1+1)$-dimensional equations) to the $(2+1)$-dimensional case. Presence of nonlocal terms in symmetries and conservation laws is the main feature of integrable $(2+1)$-dimensional equations. We have introduced a concept of *quasi-local functions* to characterize nonlocalities. We have found a few first integrability conditions for a class of scalar equations in terms of quasi-local functions and have demonstrated that they are suitable for testing integrability.

1. Introduction

Integrable nonlinear equations have many applications in physics and mathematics and are interesting by themselves. There is a rich theory of such equations which is mostly devoted to the problem of integration and to the study of the underlying algebraic and analytic structures. It is a very challenging problem to establish whether a given equation is integrable or not, whether the powerful machinery developed for integrable equations can be applied to an equation of our particular interest in a concrete problem. There are a few approaches aimed at tackling this problem. Among them are: the approach based on the Painlevé conjecture, perturbative analyses of almost-linear or almost-integrable equations, approaches based on the existence of higher symmetries and conservation laws. Possibly the most advanced one is the symmetry approach.

The symmetry approach, suitable for $(1+1)$-dimensional nonlinear partial differential equations and difference differential equations, has been created and developed during the last 18 years [1–7]. It has proved to be a powerful tool for testing the integrability and solving the classification problem for integrable equations. In this paper we are trying to extend this theory to the $(2+1)$-dimensional case. The main feature of integrable equations in $(2+1)$ dimensions is that the equations themselves, their higher symmetries and conservation laws are non-local, and this becomes the main obstacle to a straightforward extension of the $(1+1)$-dimensional approach, which is based on the concept of locality. To overcome this problem, a new concept of *quasi-local functions*, which is a natural generalization of local functions, is introduced. All known integrable equations and their hierarchies of symmetries and conservation laws can be described in terms of quasi-local functions. This observation will be exploited for a generalization of the symmetry approach to multi-dimensions, creating integrability tests, and in the near future for a classification

0305-4470/98/316707+09$19.50 © 1998 IOP Publishing Ltd

of the most important types of equations. We have found a few integrability conditions for a class of scalar equations in terms of quasi-local functions and have demonstrated that the conditions obtained are suitable for testing integrability.

2. Formal symmetries and conservation laws in the (1 + 1)-dimensional case

In the $(1+1)$-dimensional case integrable partial differential equations (PDEs)† or systems of PDEs

$$u_t = K \tag{1}$$

possess a hierarchy of higher symmetries

$$u_{t_n} = K_n \qquad n = 0, 1, 2, \ldots \tag{2}$$

and this property can be taken as a definition of integrability. Here $K, K_n \in \mathcal{F}$ where $\mathcal{F}$ is a differential field of complex (or real) valued functions of $u = u(x,t)$ and its x-derivatives. Each function from this field depends on a finite number of variables $u_0 = u$, $u_1 = u_x$, $u_2 = u_{xx}, \ldots$, which we call the dynamical variables. With $\mathcal{F}$ we can take the algebraic closure of the field of meromorphic functions $\mathrm{Alg}(\mathbb{C}\langle\langle\mathcal{U}\rangle\rangle)$, where $\mathcal{U} = \{u_k; k = 0, 1, 2, \ldots,\}$ is the set of dynamical variables. In many applications the functions K, K_n are simply polynomials. Partial derivations ∂_x, ∂_t and ∂_{t_n} are represented in $\mathcal{F}$ by the operators

$$D = \sum_{k\geqslant 0} u_{k+1}\frac{\partial}{\partial u_k} \qquad \frac{\mathrm{d}}{\mathrm{d}t} = \sum_{k\geqslant 0} D^k(K)\frac{\partial}{\partial u_k} \qquad \frac{\mathrm{d}}{\mathrm{d}t_n} = \sum_{k\geqslant 0} D^k(K_n)\frac{\partial}{\partial u_k}$$

respectively.

Equations (2) define the symmetries of (1) if (see, for example, [6, 8])

$$[K, K_n] \stackrel{\text{def}}{=} K_*(K_n) - K_{n*}(K) = 0 \tag{3}$$

where $*$ denotes the Frechét derivative. The Frechét derivative is a linear differential operator which is assigned to any function $f \in \mathcal{F}$

$$f \to f_* \stackrel{\text{def}}{=} \sum_{k\geqslant 0} \frac{\partial f}{\partial u_k} D^k. \tag{4}$$

Orders of equation (1) and symmetries (2) are defined as the degrees of the corresponding Frechét derivatives $\mathrm{ord}K = \mathrm{deg}K_*$ and $\mathrm{ord}K_n = \mathrm{deg}K_{n*}$. We say that K_n defines a higher (or non-classical) symmetry if $\mathrm{ord}K_n > 1$.

A function $\rho \in \mathcal{F}$ is called a density of a conservation law of (1) if there exists $\sigma \in \mathcal{F}$ such that

$$\rho_t = D(\sigma). \tag{5}$$

If we are not interested in a particular form of σ we write $\rho_t \in D\mathcal{F}$, where $D\mathcal{F}$ is the image of the field $\mathcal{F}$ under the action of the operator D (i.e. $D\mathcal{F} = \{D(f); f \in \mathcal{F}\}$).

Relation (5) is obviously satisfied if $\rho = D(h)$, where $h \in \mathcal{F}$. In this case $\sigma = h_t \in \mathcal{F}$. Such densities are called trivial. Two conserved densities ρ_1, ρ_2 are considered as equivalent ($\rho_1 \simeq \rho_2$) if the difference $\rho_{12} = \rho_1 - \rho_2$ is a trivial density ($\rho_{12} \simeq 0$, i.e. $\rho_{12} \in D\mathcal{F}$). By

† For simplicity here we restrict ourselves to the consideration of scalar equations, a generalization for the vector case can be found in [4–6].

the order of a conserved density ρ we shall mean the degree of the differential operator $R = (\delta\rho/\delta u)_*$, where the variational derivative is defined as

$$\frac{\delta f}{\delta u} \stackrel{\text{def}}{=} \sum_{k\geqslant 0}(-D)^k \frac{\partial f}{\partial u_k} \qquad \forall f \in \mathcal{F}. \tag{6}$$

The variational derivative has the following useful properties [8]

$$\frac{\delta D(f)}{\delta u} = 0 \qquad \frac{\mathrm{d}}{\mathrm{d}t}\frac{\delta f}{\delta u} = \frac{\delta}{\delta u}\frac{\mathrm{d}f}{\mathrm{d}t} - K_*^{\dagger}\frac{\delta f}{\delta u} \qquad \forall f \in \mathcal{F} \tag{7}$$

($A^\dagger$ denotes the formally adjoint operator: if $A = \sum a_k D^k$ then $A^\dagger = \sum (-D)^k \cdot a_k$, and $\cdot$ is the usual operator multiplication). Moreover, if $\delta f/\delta u = 0$, $f \in \mathcal{F}$, then $f \in D\mathcal{F} + \mathbb{C}$ (the Gelfand–Manin–Shubin theorem [9]). Thus, a conserved density is non-trivial if its variational derivative does not vanish. Only non-trivial conserved densities make sense.

Let us denote by $\mathcal{R}\{D\}$ a ring of formal pseudodifferential operators (or, shorter, formal operators) of the form

$$A = a_p D^p + a_{p-1}D^{p-1} + \cdots + a_0 + a_{-1}D^{-1} + \cdots$$

with coefficients $a_k \in \mathcal{F}$. The product in $\mathcal{R}\{D\}$ is uniquely defined by

$$b_k D^k \cdot a_m D^m = b_k a_m D^{k+m} + \binom{k}{1} b_k D(a_m) D^{k+m-1} + \binom{k}{2} b_k D^2(a_m) D^{k+m-2} + \binom{k}{3} b_k D^3(a_m) D^{k+m-3} + \binom{k}{4} b_k D^4(a_m) D^{k+m-4} + \cdots \tag{8}$$

where

$$\binom{k}{m} = \frac{k(k-1)(k-2)\dots(k-m+1)}{m!}.$$

We denote by $\mathcal{R}\{D\}_+$ the subring of differential operators $\mathcal{R}\{D\}_+ \subset \mathcal{R}\{D\}$. For example, if $f \in \mathcal{F}$, then the Frechét derivative f_* does belong to $\mathcal{R}\{D\}_+$. In the scalar case, $\mathcal{R}\{D\}$ is a skew-field because for each operator $A \in \mathcal{F}$ the inverse operator A^{-1} such that $AA^{-1} = A^{-1}A = 1$ exists and is uniquely defined. Moreover, fractional powers $A^{k/p}$ of a formal operator A, $p = \deg(A)$, are well defined.

A formal operator $L \in \mathcal{R}\{D\}$

$$L = l_p D^p + l_{p-1}D^{p-1} + \cdots + l_0 + l_{-1}D^{-1} + \cdots \qquad l_k \in \mathcal{F}$$

is called a formal symmetry of order N of equation (1) if it satisfies the following inequality [3, 4, 6]

$$\deg(L_t - [K_*, L]) \leqslant \deg(K_*) + \deg(L) - N. \tag{9}$$

Here the bracket $[A, B] = A \cdot B - B \cdot A$ denotes the usual commutator of (formal pseudodifferential) operators. This definition can be easily motivated. Indeed, assuming that K_n is a symmetry of order $N > 1$ and taking the Frechét derivative of equation (3), we get $\partial_t K_{n*} - [K_*, K_{n*}] = \partial_{t_n} K_*$. If we denote $L = K_n$ and estimate the degree of the left-hand side, then we arrive at the inequality (9). Thus, the existence of a symmetry of order N implies the existence of a formal symmetry of the same order. In contrast to higher symmetries, the conditions of existence of formal symmetries can be easily obtained in terms of K and its derivatives and analysed. These conditions are not influenced by lacunae in the hierarchy of symmetries and are invariant under changes of variables. If equation (1) has an infinite hierarchy of symmetries of increasing order, then equation

$$L_t = [K_*, L] \tag{10}$$

has a non-trivial solution $L \in \mathcal{R}\{D\}$ (see [3, 4, 6]).

A formal operator $R \in \mathcal{R}\{D\}$ is called a formal conservation law of order N if it satisfies the following inequality [3, 4, 6]

$$\deg(R_t + R \cdot K_* + K_*^\dagger \cdot R) \leqslant \deg(R) + \deg(K_*) - N. \tag{11}$$

In particular, if $\rho \in \mathcal{F}$ is a conserved density of order $N > m = \operatorname{ord} K$, then $R = (\delta\rho/\delta u)_*$ is a formal conservation law of order $N - m$. If equation (1) has an infinite hierarchy of conservation laws of increasing order, then it has a formal conservation law of infinite order, i.e. the following formal operator equation

$$R_t + R \cdot K_* + K_*^\dagger \cdot R = 0 \tag{12}$$

has a non-trivial solution $R \in \mathcal{R}\{D\}$ [3]. If equation (1) has two conservation laws with densities ρ_1, ρ_2 of orders $m < N_1 < N_2$, then it has formal conservation laws $R_k = (\delta\rho_k/\delta u)_*$ of orders $N_k - m$ and a formal symmetry $L = R_1^{-1} R_2$ of order $N = N_1 - m$ [3, 4, 6].

The solvability conditions of equation (10) can be represented in the form of an infinite sequence of conservation laws $\rho_{kt} = D(\sigma_k)$, $k = -1, 0, 1, 2, \ldots$, (the *canonical conservation laws*) of the original equation (1). For example, the first condition can be written as $\rho_{-1,t} \in D\mathcal{F}$, where $\rho_{-1} = (\partial K/\partial u_m)^{-1/m}$ and $m = \operatorname{ord}(K)$. This condition is equivalent to the existence of a formal symmetry of order $m + 1$ (a formal symmetry of order m always exists because equation (1) is a symmetry for itself). Existence of first p canonical conservation laws (i.e. the fact that $\partial_t\rho_{-1}, \partial_t\rho_0, \ldots, \partial_t\rho_{p-2} \in D\mathcal{F}$) is equivalent to the existence of a formal symmetry of order $m + p$.

Let us consider an equation of the form

$$u_t = u_{xxx} + F(u, u_x) \tag{13}$$

where $F(u_0, u_1)$ is a differentiable function of its variables. Then

$$\rho_{-1} = 1 \qquad \rho_0 = 0 \qquad \rho_1 = \frac{\partial F}{\partial u_1} \qquad \rho_2 = \frac{\partial F}{\partial u} \qquad \rho_3 = \sigma_1 = D^{-1}(\rho_{1t}), \ldots. \tag{14}$$

Existence of a formal conservation law can be expressed in the form of a sequence of the *canonical potentials*, i.e. functions $\phi_k \in D\mathcal{F}$, $k = 0, 1, 2, \ldots$. In the case of equation (13), these conditions may be reduced to the requirement that all even canonical densities are trivial: $\rho_{2n} \in D\mathcal{F}$, $n = 1, 2, \ldots$.

For a given PDE it is very easy to check the conditions for the existence of a formal symmetry and/or a formal conservation law (it is sufficient to check that $\delta\rho_{kt}/\delta u = \delta\phi_k/\delta u = 0$). They are so restrictive that, in practice, if a PDE satisfies a few first conditions (the number of the conditions required depends on the order of the PDE), then it is integrable. Moreover, in many cases these conditions enable us to find complete lists of integrable equations and classify them [3–7].

3. Concept of quasi-local functions

The whole construction of the symmetry approach is based upon the concept of *local functions*, i.e. functions belonging to the field $\mathcal{F}$. Symmetries, conservation laws, coefficients of formal symmetries and formal conservation laws are assumed to belong to $\mathcal{F}$. Sometimes in the $(1 + 1)$-dimensional case a simple (non-invertible in the classical sense) change of variables of the form $v = \phi(u_1)$ could violate the local structure in the sense that symmetries and conservation laws become dependent on the primitives (indefinite integrals), i.e. they become non-local while equations remain integrable by the inverse transform method or via Cole–Hopf type transformations. Extending the field $\mathcal{F}$ by just the adjunction of one primitive (or a finite number of primitives in some cases) and taking the closure we can come back to exactly the same scheme (within the extended field). Such cases are called weakly non-local and can be treated via the symmetry approach after a simple extension of the field [10].

In the $(2+1)$-dimensional case we have a different picture. If we look at the structure of higher symmetries and conservation laws of integrable equations we discover that they are non-local (even equations themselves being rewritten in the evolutionary form are non-local as a rule). The higher symmetry we take the more complex structure of non-local terms we find. Consider, for example, the hierarchy of symmetries of the Kadomtsev–Petviashvili equation

$$
\begin{aligned}
u_{t_1} &= u_x \qquad u_{t_2} = u_y \qquad u_{t_3} = u_{xxx} + 6uu_x - 3D^{-1}(u_{yy}) \\
u_{t_4} &= u_{xxy} - D^{-2}u_{yyy} + 2u_x D^{-1}(u_y) + 4uu_y \\
u_{t_5} &= u_{xxxxx} - 10u_{xyy} + 5D^{-3}u_{yyyy} + 20u_x u_{xx} + 10uu_{xxx} + 30u^2u_x - 10u_x D^{-2}u_{yy} \\
&\quad -20uD^{-1}(u_{yy}) - 20u_y D^{-1}(u_y) - 10D^{-1}(uu_y)_y \\
u_{t_6} &= u_{xxxxy} - \tfrac{10}{3}u_{yyy} + D^{-4}u_{yyyyy} + 16u^2u_y + 12u_x u_{xy} + 8u_y u_{xx} + 8uu_{xxy} \\
&\quad +12uu_x D^{-1}(u_y) + 2u_{xxx}D^{-1}(u_y) - 6D^{-1}(u_y)D^{-1}(u_{yy}) \\
&\quad -4u_y D^{-2}(u_{yy}) - 4uD^{-2}(u_{yyy}) - 2u_x D^{-3}(u_{yyy}) + 4u_x D^{-1}(uu_y) \\
&\quad -2D^{-1}(uD^{-1}(u_{yy}))_y \ldots
\end{aligned}
$$

where D^{-1} stands for indefinite integral $D^{-1}f = \int^x f(x', y)\,\mathrm{d}x'$, and D^{-2} corresponds to the nesting integrations $D^{-2}f = \int^x \int^{x'} f(x'', y)\,\mathrm{d}x''\,\mathrm{d}x'$, etc. It can be easily shown that there is not any finite extension of the field of the local functions which would contain the whole hierarchy of symmetries (or conservation laws). In the case of the Benney–Roskes–Davey–Stewartson equation

$$
\begin{cases}
u_t = \alpha(u_{xx} + 2uD_y^{-1}(uv)_x) + \beta(u_{yy} + 2uD_x^{-1}(uv)_y) \\
-v_t = \alpha(v_{xx} + 2vD_y^{-1}(uv)_x) + \beta(v_{yy} + 2vD_x^{-1}(uv)_y)
\end{cases}
\qquad \alpha, \beta \in \mathbb{C} \tag{15}
$$

we are forced to consider nested primitives in y as well ($D_y^{-1}f = \int^y f(x, y')\,\mathrm{d}y'$) and nested primitives for higher flows. A similar picture takes place for all known integrable $(2+1)$-dimensional hierarchies.

A trivial idea to extend the differential field $\mathcal{F}$ by the adjunction of all possible nested primitives ($\mathcal{F} \to \mathcal{F}(D_x^{-1}, D_y^{-1})$) seems not to be very fruitful. Doing so, we would find that any equation (even known to be non-integrable) possesses a formal symmetry, and even the concept of conservation laws would be lost because any function $f \in \mathcal{F}(D_x^{-1}, D_y^{-1})$ is a total derivative $f \in D\mathcal{F}(D_x^{-1}, D_y^{-1})$ in such a field, i.e. all conservation laws are trivial ($\mathcal{F}(D_x^{-1}, D_y^{-1})/D\mathcal{F}(D_x^{-1}, D_y^{-1}) = \mathbb{C}$).

Our generalization of the symmetry approach to the $(2+1)$-dimensional case is based on an observation that operators D_x^{-1}, D_y^{-1} never appear alone! If we introduce operators

$$
\Theta = D_x^{-1}D_y \qquad \Theta^{-1} = D_y^{-1}D_x
$$

then many classes of equations† and their hierarchies of higher symmetries and conservation laws can be rewritten without D_x^{-1}, D_y^{-1}.

† A slight generalization of the operator Θ is needed for certain classes of equations, like the Darboux (or resonance three wave) system of equations, but for simplicity here we shall restrict ourselves to a subclass of equations whose hierarchies can be written in terms of Θ and Θ^{-1}.

The previously mentioned Kadomtsev–Petviashvili (KP) equation and (15) together with the hierarchies of higher symmetries and conservation laws can be easily rewritten without D_x^{-1} and D_y^{-1}, using the operators Θ and Θ^{-1}. Many other known integrable equations including the modified KP equation

$$u_t = u_{xxx} - 6u^2 u_x + 6u_x \Theta u + 3\Theta^2 u_x \tag{16}$$

the (2 + 1)-dimensional generalization of the Degasperis–Magri–Pirani–Soliani equation found in [11] (2D DMPS equation)

$$u_t = u_{xxx} - \tfrac{1}{2}u_x^3 + 3\,\mathrm{e}^{-u}\Theta^2(\mathrm{e}^u)_x - \tfrac{3}{2}(\mathrm{e}^{2u} + \mathrm{e}^{-2u}(\Theta\,\mathrm{e}^u)^2 - 2\Theta(\mathrm{e}^{-u}\Theta\,\mathrm{e}^u))u_x \tag{17}$$

and the Nizhnik–Veselov–Novikov equation ($\alpha, \beta \in \mathbb{C}$)

$$u_t = \alpha(u_{xxx} + 3u_x\Theta^{-1}(u^2) + 3u\Theta^{-1}(uu_x)) + \beta(u_{yyy} + 3u_y\Theta(u^2) + 3u\Theta(uu_y)) \tag{18}$$

fall into that class. Moreover, using the operator Θ we can eliminate all y-derivatives ($D_y = \Theta D_x$).

In all these listed examples, the hierarchies of symmetries and conservation laws belong to a field $\mathcal{F}(\Theta) \supset \mathcal{F}$, which we call the field of quasi-local functions. To define this field, let us consider a sequence of extensions of the original field $\mathcal{F}$. Let $\Theta\mathcal{F} = \{\Theta f;\ f \in \mathcal{F}\}$, $\Theta^{-1}\mathcal{F} = \{\Theta^{-1} f;\ f \in \mathcal{F}\}$ and $\mathcal{F}_0(\Theta) = \mathcal{F}$. The field $\mathcal{F}_k(\Theta)$ is defined as the closure (i.e. the field and the algebraic closure) of the union $\mathcal{F}_{k-1}(\Theta) \cup \Theta\mathcal{F}_{k-1}(\Theta) \cup \Theta^{-1}\mathcal{F}_{k-1}(\Theta)$. We have a filtration of fields $\mathcal{F} = \mathcal{F}_0(\Theta) \subset \mathcal{F}_1(\Theta) \subset \mathcal{F}_2(\Theta) \subset \ldots$, and $\mathcal{F}(\Theta) = \lim_{k\to\infty} \mathcal{F}_k(\Theta)$. Each element of $\mathcal{F}_k(\Theta)$ is a function of a finite number of arguments, and each argument belongs to $\mathcal{F}_{k-1}(\Theta)$ or $\Theta\mathcal{F}_{k-1}(\Theta)$ or $\Theta^{-1}\mathcal{F}_{k-1}(\Theta)$. The index k in $\mathcal{F}_k(\Theta)$ indicates the maximal depth of a nesting of the operators $\Theta^{\pm 1}$ in expressions. The derivation D_y in $\mathcal{F}(\Theta)$ is represented by $D_y = \Theta D_x$.

Example. If we eliminate all the y derivatives in the KP hierarchy (using $D_y = \Theta D_x$), we find that $u_{t_k} \in \mathcal{F}_{k-1}(\Theta)$. Similarly, the right-hand side functions of the modified KP and 2D DMPS equations belong to $\mathcal{F}_2(\Theta)$, the right-hand side functions of the Benney–Roskes–Davey–Stewartson equation belong to the extension $\mathcal{F}_2(\Theta)$ of the differential field $\mathcal{F}$ with two indeterminates u and v.

In the field $\mathcal{F}(\Theta)$ the notion of conservation laws makes sense, since the factor $\mathcal{F}(\Theta)/D\mathcal{F}(\Theta)$ is non-trivial. A function $\rho \in \mathcal{F}(\Theta)$ is called a density of a conservation law if there exists $\sigma \in \mathcal{F}(\Theta)$ such that $\rho_t = D(\sigma)$. Usual 'gradient' definitions ($\rho_t = D_x(j_x) + D_y(j_y)$) can always be transformed into this form with $\sigma = j_x + \Theta(j_y)$. Moreover, if ρ is a density of a conservation law, so is $\Theta^k(\rho)$ for any k.

In conclusion of this section we would like to mention that the linearizations of integrable (2+1)-dimensional equations belong to $\mathcal{F}(\Theta)$. This fact follows from the results of Zakharov and Shulmann [12]. They have shown that the dispersion laws of integrable equations must satisfy certain constraints which imply that the linear part of such equations can always be written in terms of quasi-local functions. Here we go much further, namely, we assert that nonlinear $(2+1)$-dimensional integrable equations, their symmetries and conservation laws can also be expressed in terms of (properly defined) quasi-local functions.

4. Integrability conditions in the (2 + 1)-dimensional case

The definition of the symmetries of (2 + 1)-dimensional equations is exactly the same as in the (1+1)-dimensional case (3), but the Frechét derivative now becomes a pseudodifferential operator

$$f \to f_* = \sum_{n\leqslant m} f_n D^n \qquad f_n = \sum_{s=-k_n}^{k_n} f_{ns}\Theta^s \qquad f_n, f_{ns} \in \mathcal{F}_{k_n}(\Theta). \tag{19}$$

Let us define a ring $\mathcal{R}\{D; \Theta\}$ of formal (pseudodifferential) operators of the form

$$A = a_p D^p + a_{p-1} D^{p-1} + \cdots + a_0 + a_{-1} D^{-1} + \cdots \tag{20}$$

where coefficients a_n are *quasi-local operators*

$$a_n = \sum_{s=-k_n}^{k_n} a_{ns}\Theta^s \qquad a_{ns} \in \mathcal{F}_{k_n}(\Theta). \tag{21}$$

A set of quasi-local operators we denote by $\mathcal{R}\{\Theta\}$. The multiplication law in $\mathcal{R}\{D; \Theta\}$ is uniquely defined by (8), $D \cdot \Theta = \Theta \cdot D$, and

$$\Theta \cdot a = a\Theta - (D(a)\Theta - \Theta(D(a)))D^{-1} + (D^2(a)\Theta - \Theta(D^2(a)))D^{-2} - \cdots. \tag{22}$$

Under this multiplication, $\mathcal{R}\{D; \Theta\}$ becomes an associative ring (later we shall usually omit the $\cdot$). The degree of A is equal to p, i.e. the maximal exponent of D. Formal operators of zero and negative degree form a subring $\mathcal{R}_-\{D; \Theta\} \in \mathcal{R}\{D; \Theta\}$. The product of quasi-local operators belongs to $\mathcal{R}_-\{D; \Theta\}$.

We shall assume that the right-hand side K, K_n of equation (1) and its higher symmetry (2) belong to $\mathcal{F}(\Theta)$, then Frechét derivatives of K, K_n belong to $\mathcal{R}\{D; \Theta\}$. As before, orders of (1) and the symmetry (2) coincide with $\deg K_*$ and $\deg K_{n*}$. The definition of a formal symmetry remains the same (see (9)) with L belonging to $\mathcal{R}\{D; \Theta\}$. It is easy to see that the existence of a symmetry K_N of order N implies the existence of a formal symmetry $L = (K_N)_*$ of the same order. As in the (1 + 1)-dimensional case, the solvability conditions of equation (10) for coefficients of L, provides us with integrability conditions for the PDE expressed in terms of the coefficients of K_*.

Conditions of existence of a formal symmetry provide us with conditions of integrability. The conditions look very similar to the one-dimensional case. As an example we shall study equations of the form

$$u_t = u_{xxx} + g(u, u_x; \Theta, \Theta^{-1}) \qquad g(u, u_x; \Theta, \Theta^{-1}) \in \mathcal{F}(\Theta) \tag{23}$$

which contains the KP, modified KP, 2D DMPS and Nizhnik–Veselov–Novikov equations, etc. The Frechét derivative of the right-hand side of the equation is of the form

$$g_* = D^3 + g_1 D + g_0 + g_{-1} D^{-1} + \cdots$$

where $g_k \in \mathcal{R}\{\Theta\}$ are quasi-local operators

$$g_k = \sum g_{km}\Theta^m \qquad g_{km} \in \mathcal{F}(\Theta).$$

In this case a first few integrability conditions have a form very similar to the one-dimensional case (14).

Proposition 1. Equation (23) has a formal symmetry:

(i) of order six iff

$$\frac{\partial g_1}{\partial t} \in [D, \mathcal{R}\{\Theta\}] \tag{24}$$

i.e. $\exists \sigma_1 = \sum_m \sigma_{1m}\Theta^m$ such that $\sum_m D(\sigma_{1m})\Theta^m = \frac{\partial g_1}{\partial t}$

(ii) of order seven iff it has a formal symmetry of order six and

$$\frac{\partial g_0}{\partial t} \in [D, \mathcal{R}\{\Theta\}] \tag{25}$$

(iii) of order eight iff it has a formal symmetry of order seven and

$$\frac{\partial}{\partial t}(\sigma_1 + 3g_{-1}) + [\sigma_1, g_1]_{-1} \in [D, \mathcal{R}\{\Theta\}]. \tag{26}$$

In (26) we denote by $[\sigma_1, g_1]_{-1}$ the coefficient at D^{-1} in the commutator

$$[\sigma_1, g_1] = [\sigma_1, g_1]_{-1}D^{-1} + [\sigma_1, g_1]_{-2}D^{-2} + [\sigma_1, g_1]_{-3}D^{-3} + \cdots$$

of quasi-local operators σ_1 and g_1.

Similar to the one-dimensional case, one can define a formal conservation law (a formal operator $R \in \mathcal{R}\{\Theta\}$ which satisfies the same equation (12)), orders of a conserved density and formal conservation law, etc. Also, one can prove that if there exists a conservation law with a density of order N, then there exists a formal conservation law of order $N - \deg K_*$.

Proposition 2. Equation (23) has a formal conservation law of order four iff

$$g_0 \in [D, \mathcal{R}\{\Theta\}]. \tag{27}$$

Similar to the one-dimensional case, it is not difficult to obtain conditions (criteria) for the existence of formal symmetries of orders nine, ten, ... and of formal conservation laws of orders five, six, ..., but they look more complicated and will not be used in this paper. The integrability conditions have the form $\varphi \in [D, \mathcal{R}\{\Theta\}]$ where φ is a given quasi-local operator. One can analyse such conditions using properly defined variational derivatives and successive application of the Frechét derivatives.

For example, in the case of the modified Kadomtsev–Petviashvili equation (16) we have

$$g_1 = -6u^2 + 6\Theta(u) + 3\Theta^2$$
$$g_0 = -12uu_1 + 6u_1\Theta \qquad g_{-1} = 0.$$

It follows from (24) that g_1 should be a density of a conservation law, and indeed $(g_1)_t = D(\sigma_1)$ where

$$\sigma_1 = 6u_1^2 - 12uu_2 + 18u^4 - 36u^2\Theta u + 18(\Theta u)^2 + 6\Theta u_2 - 18\Theta^2(u^2) + 18\Theta^3 u.$$

Conditions (25) and (27) are satisfied, since the coefficient g_0 is a total derivative itself

$$-12uu_1 + 6u_1\Theta = [D, -6u^2 + 6u\Theta].$$

Condition (26) gives the next conservation law of order two with the density σ_1 (the term $[\sigma_1, g_1]_{-1} = 6D(\sigma_1)\Theta^2 - 6D(\Theta(\sigma_1))\Theta$ is a total derivative itself and therefore can be omitted).

In the case of the 2D DMPS equation (17), the first condition (24) gives a conservation law of order 2, the second condition is trivially satisfied because the corresponding g_0 is a total derivative (in accordance with proposition 2), condition (26) gives us a conservation law of order 4.

Thus, if we start with an integrable equation, then the solvability conditions for a formal symmetry provide us with conservation laws of the equation. Like the $(1+1)$-dimensional case, these conditions may serve as a test for integrability and help to isolate integrable cases for a given equation. As an example we consider the following equation (one more (2 + 1)-dimensional generalization of the Korteweg–de-Vries equation)

$$u_t = u_{xxx} + u\Theta^{-1}u_x + \lambda u_x \Theta^{-1} u \tag{28}$$

where λ is a constant parameter. The Frechét derivative of the right-hand side of this equation is of the form $F_* = D^3 + g_1 D + g_0$ with

$$g_1 = u\Theta^{-1} + \lambda\Theta^{-1}(u) \qquad g_0 = \Theta^{-1}(u_x) + \lambda u_x \Theta^{-1}. \tag{29}$$

Condition (24) is obviously satisfied for any λ with

$$\sigma_1 = \sigma\Theta^{-1} + \lambda\Theta^{-1}(\sigma) \qquad \sigma = u_{xx} + u\Theta^{-1}(u) + \frac{\lambda - 1}{2}\Theta((\Theta^{-1}(u))^2). \tag{30}$$

The coefficient g_0 is a total derivative for any λ, therefore the conditions (25) and (27) are satisfied. It follows from the condition (26) and (30) that σ should be a density of a conservation law. The first term u_{xx} is a trivial density (it is a total derivative), the second term $u\Theta^{-1}(u)$ is a conserved density for any λ, but the last term gives the condition $\lambda = 1$, because $\partial_t(\Theta^{-1}(u))^2$ does not belong to $D\mathcal{F}(\Theta)$ for any λ. Thus $\lambda = 1$ is a necessary condition for integrability of equation (28). The same condition was obtained by the Painlevé method in [13]. Moreover, equation (28) is known to be integrable if $\lambda = 1$ [14].

Acknowledgments

We would like to thank the NATO-Royal Society Fellowship Programme for supporting RYs visit to Leeds. The basic concept of quasi-local functions was discussed at the NEEDS97 Workshop and especially we would like to thank B Konopelchenko, M D Kruskal, S V Manakov and V Zakharov for their useful remarks. We are also grateful to P Clarkson, W Crawley-Boevey, G Dales, A Fordy, J McConnel, C Robson and A Shabat for discussions on various aspects of the paper.

References

[1] Zhiber A V and Shabat A B 1979 *Dokl. Acad. Nauk.* **247** 1104 (Engl. Transl. *Sov. Math.–Dokl.*)
[2] Ibragimov N H and Shabat A B 1980 *Funk. Analiz.* **14** 79
[3] Sokolov V V and Shabat A B 1984 *Sov. Sci. Rev.* C **4** 221
[4] Mikhailov A V, Shabat A B and Yamilov R I 1987 *Usp. Mat. Nauk.* **42** 3 (Engl. Transl. 1987 *Russian Math. Surveys* **42** 1)
[5] Mikhailov A V, Shabat A B and Yamilov R I 1988 *Commun. Math. Phys.* **115** 1
[6] Mikhailov A V, Shabat A B and Sokolov V V 1991 *What is Integrability?* ed V E Zakharov (Berlin: Springer) p 115
[7] Shabat A B and Mikhailov A V 1993 *Important Developments in Soliton Theory* ed A Fokas and V E Zakharov (Berlin: Springer) p 355
[8] Olver P J 1986 *Application of Lie Groups to Differential Equations* (Berlin: Springer)
[9] Gelfand I M, Manin Y I and Shubin M 1976 *Funk. Analiz.* **10** 274
[10] Mukminov F Kh and Sokolov V V 1987 *Mat. Sbor.* **133** 392
[11] Yamilov R I 1993 *Phys. Lett.* **173A** 53
[12] Zakharov V E and Shulman E 1980 *Physica* **10D** 192
[13] Weiss J, Tabor M and Carnevale G 1983 *J. Math. Phys.* **24** 522
[14] Boiti M, Leon J J-P, Manna M and Pempinelli F 1986 *Inverse Problems* **2** 271